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The Leslie Matrix Model and Its Applications in Actuarial Science

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The Leslie Matrix Model and Its Applications in Actuarial Science

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<u>Dedication</u>

To my family, whose support has guided me every step of the way, and to Robert, who may not be family by blood but will always feel like family to me. His invaluable advice and inspiration have been a guiding light throughout this journey.

Acknowledgments

I would like to express my deepest gratitude to my project director, Dr. Brent Dozier, Associate Professor of Mathematics. As my favorite professor and someone who shares my passion for the actuarial field, his invaluable guidance, mentorship, and support have been instrumental throughout this thesis.

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Abstract

This thesis explores the Leslie matrix model, a mathematical framework that analyzes population dynamics, and its applications in actuarial science. This paper demonstrates how the Leslie matrix projects population structure over time based on fertility and survival rates, starting with fundamental ideas like matrix multiplication, eigenvalues, and eigenvectors. These dynamics are illustrated through a case study of a fictitious female population in Buenos Aires. The dominating eigenvalue, estimated to be $\lambda \approx 1.2512747$, governs the model's long-term behaviors, which include stable age distributions and exponential population growth, as revealed via iterative matrix multiplication and eigenvalue analysis. Additionally, the study emphasizes how early growth ratios show oscillations before to reaching this long-term growth rate. Applications in actuarial contexts are discussed, demonstrating how such models can help with pension sustainability, life insurance planning, and demographic risk assessments. The thesis concludes that the Leslie matrix is an effective instrument for anticipating population changes and guiding actuarial decision-making in the face of demographic uncertainty.

Understanding the Basics

Before diving into the specifics of the Leslie Matrix Model, it is critical to lay a solid foundation by reviewing fundamental mathematical concepts. This chapter covers matrices, vectors, and their interactions, which are crucial building blocks for comprehending more complex population models. By presenting these concepts clearly and simply, readers with limited mathematics background will be able to follow the paper.

What is a Matrix?

Similar to a table, a matrix is a straightforward method of arranging numbers in rows and columns. Here is an example of a matrix:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Three rows, or horizontal lines of numbers, and three columns, or vertical lines of numbers, make up this matrix. This is known as a "three-by-three" or 3x3 matrix. Data or relationships are frequently represented by the numbers inside the matrix.

In simpler words, a matrix can be thought of as a box that contains data. It facilitates computations and the handling of large amounts of data.

What is a Vector?

A vector is a list of numbers that are often displayed as a single row or column. Here is an example of a column vector:

$$\begin{bmatrix} 200 \\ 120 \\ 80 \end{bmatrix}$$

Vectors are crucial because they make it possible to arrange data in a way that makes computations simple.

Matrix Multiplication

Matrix multiplication is simply a method of mixing the numbers in a vector and a matrix. Here is a step-by-step explanation:

1. Select a row from the matrix and the corresponding column from the vector.
2. Multiply the numbers that match.
3. To obtain a new number, add those outcomes.

For example, suppose we multiply the $2x2$ matrix:

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

with this vector

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This is how we compute $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} . \begin{bmatrix} 5 \\ 6 \end{bmatrix}$:

Multiply the matrix's first row by the vector:

$$(1 * 5) + (3 * 6) = 5 + 18 = 23$$

Multiply the matrix's second row by the vector:

$$(2 * 5) + (4 * 6) = 10 + 24 = 34$$

A new vector is the outcome:

$$\begin{bmatrix} 23 \\ 34 \end{bmatrix}$$

What are Eigenvalues and Eigenvectors?

A nonzero vector v is called an eigenvector of A if there exists a real number (λ) such that $Av = \lambda v$. The number λ is the eigenvalue corresponding to v. Eigenvalues are the solutions to the characteristic equation, which comes from setting the determinant ($A - \lambda I$) equal to zero. Each eigenvalue has an associated eigenspace consisting of all nonzero vectors v that satisfy the equation $Av = \lambda v$ (Lay 2012, Strang 2016).

Imagine a space experiencing a transformation, such as compression, rotation, or stretching. This transformation is represented by a square matrix A in the language of linear algebra. An eigenvector of A is a special nonzero vector v that does not change direction when A is applied; instead, it gets scaled by a number (λ). The eigenvalue λ is the scale factor applied to v.

In other words:

- Eigenvector (v) is the direction that only changes in length under the transformation by A.
- Eigenvalue (λ) is the amount by which that length changes.

To make these ideas easier to understand, we illustrate them with an example. Consider a simple 2×2 matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

We want to find nonzero vectors v such that when A acts on v, the result is merely a scaled version of v. Suppose $v = \begin{bmatrix} x \\ y \end{bmatrix}$, then:

$$Av = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

We seek a scalar λ satisfying:

$$\begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

This gives two equations:

$$x + 2y = \lambda x$$

$$2x + y = \lambda y$$

For nontrivial solutions ($v \neq 0$), we require that the determinant of $A - \lambda I$ equals zero. Here, I is the identity matrix, so we set up:

$$det\ (A - \lambda I) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix} = 0$$

Calculating the determinant:

$$(1 - \lambda)^2 - 4 = 0$$

Solving for λ:

$$[(1-\lambda)(1-\lambda)]-4=0$$

$$1-\lambda-\lambda+\lambda^2-4=0$$

$$\lambda^2-2\lambda-3=0$$

Factoring:

$$(\lambda-3)(\lambda+1)=0$$

The solutions are:

$$\lambda_1=3\,,\lambda_2=-1$$

Next, we find the corresponding eigenvectors. For $\lambda=3$, we solve:

$$\begin{bmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1-3 & 2\\ 2 & 1-3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

From the first row, $-2x+2y=0 \Rightarrow -2x=-2y \Rightarrow$ x = y

So one possible solution is x $=1$ and y $=1$. An eigenvector for $\lambda=3$ is therefore:

$$v_1=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Because eigenvectors are only determined up to scale, any nonzero multiple of v_1 is equally valid.

We find the corresponding eigenvectors. For $\lambda = -1$, we solve:

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row, $2x + 2y = 0 \Rightarrow 2x = -2y \Rightarrow -x = y$

So one possible solution is $x = 1$ and $y = -1$. An eigenvector for $\lambda = 3$ is therefore:

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Because eigenvectors are only determined up to scale, any nonzero multiple of v_2 is equally valid.

Eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ tell us how the matrix A scales its eigenvectors:

- If you apply A to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, it gets scaled by 3.
- If you apply A to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, it gets scaled by -1 (which flips the direction)

- Eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ represent special directions where the transformation by A is purely a scaling, without any rotation off that direction.

Eigenvalues and eigenvectors are critical for understanding how linear transformations (represented by matrices) affect vectors.

Brief Introduction to Actuarial Science

Actuarial science is a discipline that uses mathematical and statistical tools to evaluate risk in businesses such as insurance, banking, and pension management (Bowers et al., 1997). Actuaries assess the possibility of future events, such as mortality rates or retirement trends, and assist businesses in developing plans to address these uncertainties (Bowers et al., 1997).

Here are a few significant areas where demographic forecasts play an important role:

- Life insurance and annuity products rely heavily on accurately forecasting life expectancy (Lee & Carter, 1992). If people live longer than expected, insurers and pension funds may face financial difficulties.
- Pension sustainability depends on balancing contributions from working-age participants with payouts to retirees (Barr & Diamond, 2008). An unanticipated increase in the number of retirees (for example, as a result of increasing longevity) can put a burden on pension funds or force policy changes.
- Insurance premium rates are based on predicted claims over time (Klugman et al., 2012). Inaccurate demographic data, such as an unforeseen baby boom or longer lifespans, can throw these estimates off, causing insurance policies to be overpriced or underpriced.

Historical Development of the Leslie Matrix Model in Population Studies

The concept of predicting population changes by using age-specific survival and fertility rates dates back to the late 19th century. Early efforts in projecting human populations can be traced to Cannan (1895), with subsequent studies by Smith and Keyfitz (1977) and Cohen (1995) documenting these initial attempts (Caswell, 2001).

The use of matrix methods to model population dynamics was independently explored and developed further by Bernardelli (1941), Lewis (1942), and Leslie (1945), with Leslie's work proving to be the most influential. Patrick Holt Leslie, a British physiologist, played a pivotal role in establishing the Leslie matrix as a core tool in population modeling (Caswell, 2001).

Leslie's journey into population studies began under unique circumstances. Initially pursuing a career in medicine, Leslie was forced to change direction due to a severe case of tuberculosis, which prevented him from continuing in the field. In 1935, he joined the Bureau of Animal Population at Oxford, working under Charles Elton, who suggested the importance of studying mortality and fertility schedules. Inspired by Elton's idea, Leslie began exploring human demographic data and searching for effective techniques to integrate survival and birth rates into a single, predictive model (Caswell, 2001).

Patrick Holt Leslie at the Bureau of Animal Population, Oxford University, about 1950.

Photograph courtesy of Sir Peter Leslie

This endeavor led to the first application of Lotka's equation for determining the intrinsic rate of increase in animal populations. Collaborating with Ranson in 1940, Leslie began developing a systematic approach to model age-specific population projections, which ultimately resulted in his innovative 1945 paper. In this work, Leslie expressed age-specific projection equations in matrix form, setting the foundation for what would later be known as the Leslie matrix. This matrix model allowed researchers to calculate future population distributions by iterating the matrix over time (Caswell, 2001).

Alongside Leslie, other researchers like Bernardelli and Lewis were also contributing to matrix-based population models. Bernardelli's 1941 paper on "Population Waves" introduced matrix methods to study fluctuations in age distribution. Unlike Leslie, Bernardelli's work focused more on population oscillations rather than long-term stability. He used a matrix model to describe oscillations in the Burmese population between 1901 and 1931, highlighting the potential for cycles within age-structured populations (Caswell, 2001).

In the post-1945 period, Leslie's work continued to gain recognition and influence. His 1948 and subsequent papers further expanded on matrix population models, covering topics like life table analysis and the application of these models to both animal and human populations. The Bureau of Animal Population, where Leslie worked until his retirement, became a center for ecological and population studies, attracting researchers who would later make significant contributions to mathematical ecology (Caswell, 2001).

Another notable figure, Leonard Lefkovitch, further advanced matrix population models by introducing classification based on developmental stages rather than chronological age. Lefkovitch's research in the 1960s focused on pest population dynamics, applying Leslie's matrix concepts to laboratory studies. His work provided insights into using matrix models for agricultural and ecological applications, proving the versatility of Leslie's foundational ideas (Caswell, 2001).

Leonard Lefkovitch around 1965, while working at the Agricultural Research Council Pest Research Laboratory. Photograph courtesy of the Agricultural Research Council Pest Research Laboratory archives.

Despite its initial mathematical complexity, the Leslie matrix model gradually gained popularity, especially as computers became more accessible. In the 1970s, plant ecologists began to adopt matrix population models, recognizing their potential for predicting demographic changes. Researchers like Sarukhan, Gadgil, Hartshorn, and Caswell applied matrix methods to study plant populations, where size was sometimes a better predictor than age (Caswell, 2001).

By the late 20th century, the Leslie matrix had become an established tool in population ecology, actuarial science, and conservation biology. Its adoption marked a shift in how scientists approached population dynamics, moving from simple life tables to sophisticated, age-structured models. The Leslie matrix remains a vital tool today, enabling researchers to analyze demographic changes and make long-term predictions for both human and wildlife populations (Caswell, 2001).

Structure and Function of the Leslie Matrix Model

The Leslie matrix model offers a structured, mathematical framework for predicting how populations change over time using age-specific fertility and survival rates (Leslie, 1945). The model is based on matrix algebra concepts and categorizes populations into distinct age groups. The model operates iteratively, applying the matrix to the population vector at each time step to project future population changes. This technique has become essential for demographic forecasting, ecological modeling, and risk analysis in actuarial science (Keyfitz & Caswell, 2005). This matrix is an age-classified projection matrix made up of fertility and survival rates. These rates dictate how the population shifts across age groups at each time step, whether that period is measured in years, months, or other units.

Assumptions of the Leslie Matrix Model

The accuracy and applicability of the Leslie matrix model are based on a number of important assumptions. First, the population is broken down into age groups, each representing people within a specific age range. The model assumes constant fertility and survival rates throughout the projection period, simplifying calculations while maintaining prediction accuracy (Pollard, 1973). Furthermore, many applications of the model concentrate primarily on females, who are directly responsible for reproduction, decreasing the complexity of demographic forecasting by focusing on the reproductive component of population growth (Keyfitz & Caswell, 2005). The model also assumes a closed population, which means that external effects like immigration and emigration are ignored until expressly included through changes (Leslie, 1945). Together, these assumptions establish a balance between simplicity and analytical strength, allowing the Leslie matrix to be a useful tool in demographic research.

Components of the Leslie Matrix Model

The Leslie matrix model is built around three key components: the population vector, fertility rates, and survival rates. These factors interact to model population dynamics throughout time, including both birth rates and age transitions (Leslie, 1945; Pollard, 1973). Understanding these components is critical for evaluating the model's conclusions and applying them effectively in predicting.

1. Population Vector

The distribution of people in various age groups at a specific moment in time is represented by the population vector $N(t)$. It has the following structure:

$$N(t) = \begin{bmatrix} n_0(t) \\ n_1(t) \\ \vdots \\ n_k(t) \end{bmatrix}$$

The number of people in the i-th age class at time t is indicated here by $n_i(t)$. There are $k + 1$ age classes (Leslie, 1945). Every vector entry represents a distinct age group, which could be a time period. For example, for $i = 0$, the corresponding age group might be 0–1 years, and for $i = 1$, the age group might be 1–2 years, and so on. The vector $N(t)$ fluctuates over time as a result of the effects of survival and fertility rates, illustrating how population structure shifts (Pollard, 1973).

For example, suppose the starting population is represented by the vector: $N(0) = \begin{bmatrix} 110 \\ 90 \\ 70 \end{bmatrix}$

This means that the youngest age group has 110 members, the middle age group has 90, and the oldest age group has 70.

2. Fertility Rates

Fertility rates are an important factor in the Leslie matrix model that affects the number of new people born into the youngest age group (Leslie, 1945). These births are calculated by multiplying the number of females in each age group by the fertility rate of that group. The reproductive potential of each age group is represented by the fertility rates, represented by F_1, F_2, $F_3, \dots, F_s$ which are positioned in the first row of the Leslie matrix (Caswell, 1989).

The number of newborns (N_0) is determined mathematically as follows:

$$N_0 = N_1 \,.\, F_1 + N_2 \,.\, F_2 + N_3 \,.\, F_3 + \cdots + N_s \,.\, F_s$$

The average number of female children produced by an individual in a given time period is represented by the fertility rate for that age class, denoted by F_i. In this case F_i is the average number of female offspring produced by a person in the i-th age class, while N_i represents the number of individuals in that age class (Caswell, 1989).

For example, if the fertility row of the matrix is:

$$[0 \quad 1.2 \quad 1.8]$$

This indicates that individuals in the first age class produce 0 offspring, second age class produce 1.2 offspring on average, and those in the third class produce 1.8 offspring.

Difference Between Fertility and Fecundity

While fertility and fecundity are often used interchangeably, they have distinct meanings in population modeling and demography. Fertility refers to the actual reproductive performance of individuals within a population. It reflects the number of offspring produced by females over a given time period and can be influenced by social, economic, and environmental factors (Keyfitz & Caswell, 2005). In the Leslie matrix, fertility rates represent this measure by quantifying how many new individuals are born into the population (Caswell, 1989).

On the other hand, fecundity is a biological indicator of a population's or individual's capability for reproduction (Keyfitz & Caswell, 2005). It refers to the greatest number of children that could be born in perfect circumstances, free from constraints like resource availability or health. Fecundity, for instance, may refer to the number of eggs that a female organism can produce, while fertility would indicate the proportion of those eggs that produce living offspring. Fertility rates, which are typically lower because of real-world issues including mortality, infertility, and resource limits, are frequently used as a practical estimate for fecundity in population research.

3. **Survival Rates:**

Survival rates determine the likelihood that people in one age group will live to the next, which determines the population's aging process (Caswell, 1989). The sub-diagonal of the Leslie matrix shows these rates. The structure of the sub-diagonal for a population with S age classes is as follows:

$$Sub - diagonal : \begin{bmatrix} \square & \square & \square & \square \\ S_0 & 0 & 0 & \cdots \\ 0 & S_1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

S_i denotes the survival probability for people in the i-th age class (Caswell, 1989). For example, if $S_0 = 0.8$ and $S_1 = 0.9$, 80% of individuals in the first age class survive to the second, whereas

90% of individuals in the second class survive to the third. These survival rates take into consideration a variety of risks, including mortality, environmental exposure, and illness. The sub-diagonal guarantees that population aging is appropriately approximated, allowing people to progress from one age class to the next in stages. Changes in survival rates can have a considerable influence on long-term population stability, with lower rates leading to population decrease and higher rates promoting growth.

How the Leslie Matrix Is Structured

To predict how a population will vary over time, the Leslie matrix takes into account both fertility and survival rates (Leslie, 1945). The matrix contains two main sections:

- The top row shows fertility rates (F_0 , F_1 , F_2 , F_3 ... F_s) which influence the amount of new people added to the youngest age group. There are $s + 1$ age classes (Caswell, 1989).
- The sub-diagonal contains survival rates (S_0 , S_1 , S_2 ... S_{s-1}), which reflect the likelihood that people in a certain age class will survive and progress to the next age class (Caswell, 1989).

All other entries are zero, indicating that population members do not jump age groups or remain in the same group (with the exception of the last group, which may include the oldest individuals).

The Leslie matrix structure is represented visually as (Caswell, 1989):

$$\begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_s \end{bmatrix}_{t+1} = \begin{bmatrix} F_0 & F_1 & F_2 & F_3 & \dots & F_s \\ S_0 & 0 & 0 & 0 & \dots & 0 \\ 0 & S_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & S_2 & 0 & \dots & 0 \\ \dots & \square & \square & \square & \square & \square \\ 0 & 0 & 0 & 0 & S_{s-1} & 0 \end{bmatrix} . \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_s \end{bmatrix}_t$$

To illustrate how the Leslie matrix is structured, consider a population divided into four age groups: 0-10 years, 10-20 years, 20-30 years, and 30-40 years. The population dynamics can be modeled using the following Leslie matrix:

$$A = \begin{bmatrix} 0 & 1.3 & 1.9 & 2.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \end{bmatrix}$$

Where:

The first row $[0, 1.3, 1.9, 2.1]$ represents fertility rates.

The sub diagonal $[0.7, 0.8, 0.9]$ represents survival rates.

- 70% of individuals aged 0-10 survive to next age group.
- 80% of individuals aged 10-20 survive to next age group.
- 90% of individuals aged 20-30 survive to next age group.

All other values are zero, ensuring individuals do not skip or remain in age groups.

Given an initial population distribution:

$$N_t = \begin{bmatrix} 1100 \\ 900 \\ 700 \\ 600 \end{bmatrix}$$

The population at the next time step is obtained by multiplying the Leslie matrix with the population vector:

$$N_{t+1} = \begin{bmatrix} 0 & 1.3 & 1.9 & 2.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \end{bmatrix} \begin{bmatrix} 1100 \\ 900 \\ 700 \\ 600 \end{bmatrix}$$

$$N_{t+1} = \begin{bmatrix} [0(1100)] + [1.3(900)] + [1.9(700)] + [2.1(600)] \\ [0.7(1100)] + [0(900)] + [0(700)] + [0(600)] \\ [0(1100)] + [0.8(900)] + [0(700)] + [0(600)] \\ [0(1100)] + [0(900)] + [0.9(700)] + [0(600)] \end{bmatrix}$$

$$N_{t+1} = \begin{bmatrix} 3760 \\ 770 \\ 720 \\ 630 \end{bmatrix}$$

This new vector represents the projected population in each age group for the next time step.

The above computation shows how each row of the Leslie matrix effects births in the youngest cohort while shifting individuals to older age groups. Figure 1 shows these transitions in a conceptual flow diagram.

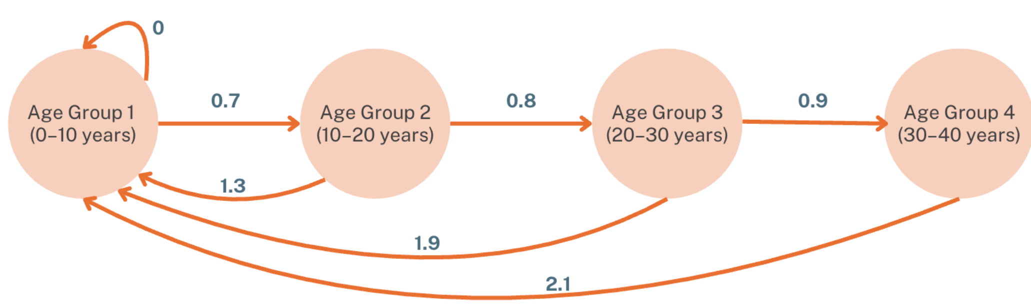


Figure 1

Arrows labeled 1.3, 1.9, and 2.1 indicate fertility contributions from older age groups into Age Group 1. The arrows labeled 0.7, 0.8, and 0.9 represent the proportion of individuals who survive and progress from one age group to the next. The “0” arrow at Age Group 1 signifies that the youngest cohort (0–10 years) does not contribute births.

Multiplication Process and Long-Term Projections

By applying a process known as iteration[1], the Leslie matrix approach enables long-term projections (Caswell, 1989). The population vector is multiplied by the matrix at each time step:

- $N_{t+1} = A * N_t$ shows the population after one time period (Keyfitz & Caswell, 2005).
- $N_{t+2} = A * (A * N_t) = A^2 * N_t$ shows the population after two time periods (Keyfitz & Caswell, 2005).
- This pattern continues, where $N_{t+n} = A^n * N_t$, predicting the population distribution over multiple years (Keyfitz & Caswell, 2005).

Matrix multiplication in this context follows the rule that each element of the resulting population vector is obtained by computing the dot product of the corresponding row in the Leslie matrix and the current population vector (Lay, 2012).

Actuaries can forecast demographic changes that have a direct influence on risk assessments, pension liabilities, life insurance policies, and healthcare funding by repeating this process over a number of time periods (Bowers et al., 1997; Billig & Gallop, 2014). Actuaries can create sustainable financial models for retirement systems and social insurance programs, for instance, by projecting changes in the working-age population and trends in lifespan. Actuaries can better predict premium adjustments, death rates, and claims frequencies by having a thorough

[1] Repeatedly applying the same mathematical operation at each time step.

understanding of future population dynamics. This helps to guarantee that financial reserves will be sufficient to cover future commitments.

Long-Term Behavior – Stable Age Distribution

Many age-structured population models, particularly the Leslie matrix model, show that the proportion of people in each age group tends to settle into a consistent pattern over time. This pattern is referred to as the steady age distribution (Leslie, 1948). Even if each age group's absolute numbers alter (increasing or decreasing), the relative percentage of each age class in the total population remains constant once stability is achieved.

When we continually project a population ahead in time (year after year) with the same survival and fertility rates, we get convergence: each age group finally settles into a fixed fraction of the total population (Keyfitz & Caswell, 2005). This indicates that if the population vector at time t is designated by $N(t)$ and expressed as proportions (rather than absolute counts), the proportions will stabilize. In other words, the ratios:

$$\frac{N_1(t)}{N_{total}(t)}, \frac{N_2(t)}{N_{total}(t)}, \dots, \frac{N_k(t)}{N_{total}(t)}$$

Each of which lies between 0 and 1 approach fixed values, where $N_i(t)$ denotes the size of the $i-th$ age class and $N_{total}(t)$ denotes the sum of all age classes at that moment. Although the precise mathematics underlying this convergence can be complex, the key point is that persistent survival and fertility trends drive the age distribution toward a stable configuration (Leslie, 1945).

Difference Between a "Stable" Distribution and a "Stable" Population

It is crucial to note that the term "stable" relates to the distribution of ages rather than the total population size. In other words:

- Stable Age Distribution: The percentage of people in each age group remains constant from one time step to the next (Leslie, 1945).
- Population Size: The total number of people in the population may still be increasing, decreasing, or remaining constant (Keyfitz & Caswell, 2005).

As a result, a population can have a constant age structure while rising fast in overall number (assuming each age class grows at the same rate) (Keyfitz & Caswell, 2005). Conversely, it can be declining while the proportions of each age group stay stable.

Intuitive Explanation

Regular Patterns of Birth and Survival: The model has been created so that, if they survive, members of each age class "move" on to the next one, while new members are added to the youngest group through births. The system eventually "balances out" since these processes operate at steady rates (Leslie, 1948).

Self-Correcting Mechanisms: If, for instance, there are momentarily "too many" people in one age group compared to others, the impact of that group on births or survival transitions aids in population redistribution in later stages. These minor imbalances eventually go away, and the age structure stabilizes (Caswell, 1989).

A Simple Illustrative Example: Buenos Aires Demographic Scenario

Consider a hypothetical population in Buenos Aires, Argentina, divided into three age groups:

- Age Group 1: Children aged 0 to 14 years.
- Age Group 2: 15 to 49 years (reproductive age)
- Age Group 3: 50+ years old (post-reproductive age)

The initial population is:

$N_0 = 120{,}000$ (Age Group 1)

$N_1 = 200{,}000$ (Age Group 2)

$N_2 = 80{,}000$ (Age Group 3)

The total population is $120{,}000 + 200{,}000 + 80{,}000 = 400{,}000$

Calculating the initial proportions:

- Age Group 1: $\frac{120{,}000}{400{,}000} = 30\%$
- Age Group 2: $\frac{200{,}000}{400{,}000} = 50\%$
- Age Group 3: $\frac{80{,}000}{400{,}000} = 20\%$

After repeatedly applying the Leslie matrix model using constant fertility and survival rates over several time steps, the age proportions adjust due to the dynamics of births and survival transitions.

To illustrate these dynamics, Figure 2 shows the Buenos Aires hypothetical population's initial distribution. Each bar shows an age group's proportion of the overall population.

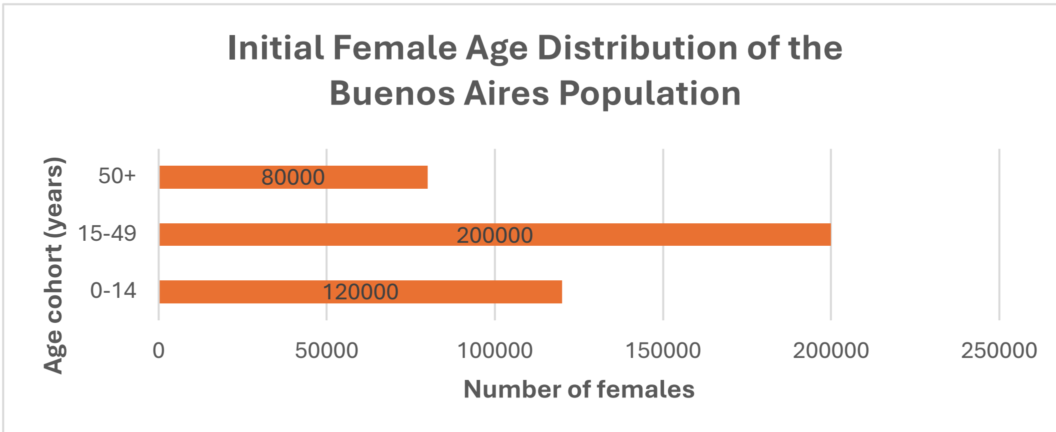


Figure 2

Eigenvalue and Eigenvector Interpretation

The dominating eigenvalue and its corresponding eigenvector are two interconnected concepts that are essential to comprehending a population's long-term behavior in the Leslie matrix model. Once equilibrium is achieved, these mathematical structures show how individuals are distributed throughout various age groups and provide information about whether a population is increasing, decreasing, or staying the same (Caswell, 1989).

Understanding the Dominant Eigenvalue (λ)

Intrinsic Rate of Increase

The dominant eigenvalue, denoted by λ, reflects the population's intrinsic rate of increase. In simple terms:

- If $\lambda > 1$, each generation is larger than the previous one, indicating a growing population (Caswell, 1989).
- If $\lambda < 1$, each generation is smaller, meaning the population is in decline (Caswell, 1989).
- If $\lambda = 1$, the population is stable, with the total number remaining constant over time (Caswell, 1989).

Interpreting the Corresponding Eigenvector

Link to the Stable Age Distribution

A particular eigenvector that gives a snapshot of the population's age distribution after it has stabilized is linked to the dominating eigenvalue. The consistent proportions of people in each age group at equilibrium are represented by this eigenvector (Caswell, 1989). To put it another way, each component of this eigenvector indicates the percentage of the population in that age group if you normalize it so that the sum of its parts equals 1 (or 100%).

Example - Hypothetical Population Projection for Buenos Aires, Argentina

To demonstrate how the Leslie matrix model works, consider a hypothetical demographic situation located in Buenos Aires, Argentina. While this example is based on true demographic factors, the data utilized is fictitious and just for demonstrative reasons. In this example, the population is separated into three age categories:

- Age Group 1: Children aged 0 to 14 years.
- Age Group 2: 15 to 49 years (reproductive age)

- Age Group 3: 50+ years old (post-reproductive age)

The initial population vector for this scenario:

$$N(0) = \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix}$$

Here:

$N_0 = 120{,}000$

$N_1 = 200{,}000$

$N_2 = 80{,}000$

The Leslie matrix for this hypothetical case is:

$$A = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}$$

In this matrix, the top row indicates fertility rates, which show that women aged 15 to 49 have an average of 1.4 offspring, whereas elderly women (50 years old and more) have a residual fertility rate of 0.5. The sub-diagonal shows survival rates: S_0 = 0.9 implies that 90% of children survive to join the reproductive age group, while S_1 = 0.85 indicates that 85% of women aged 15 to 49 live to age 50+. Figure 3 below shows the correlations captured by the Leslie matrix and helps to visualize the transitions between age groups, as well as the flow of fertility and survival. The arrows represent the movement of age groups and their contributions to the newborn population, with the corresponding fertility and survival values.

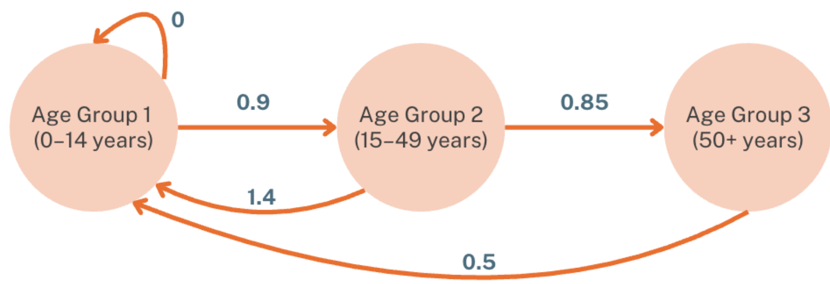


Figure 3

Projecting the Next Time Step

We compute the new population vector $N(1)$ by multiplying the Leslie matrix by the initial population vector in order to anticipate the population one time step ahead:

$$N\,(1) = A * N(0)$$

$$N\,(1) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix} . \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix}$$

Performing the calculations:

Newborns (0–14 years):

$$N_0 = \ (0)\,.\,(120{,}000) + (1.4)\,.\,(200{,}000) + (0.5)\,.\,(80{,}000)$$

$$N_0 = 0 + 280{,}000 + 40{,}000$$

$$N_0 = 320{,}000$$

Survivors from age group 0–14 to 15–49:

$$N_1 = (0.9).(120{,}000) + (0).(200{,}000) + (0).(80{,}000)$$

$$N_1 = 108{,}000 + 0 + 0$$

$$N_1 = 108{,}000$$

Survivors from age group 15–49 to 50+:

$$N_2 = (0).(120{,}000) + (0.85).(200{,}000) + (0).(80{,}000)$$

$$N_2 = 0 + 170{,}000 + 0$$

$$N_2 = 170{,}000$$

Thus, the new population vector is:

$$N\ (1) = \begin{bmatrix} 320{,}000 \\ 108{,}000 \\ 170{,}000 \end{bmatrix}$$

Interpretation After One Time Steps

Buenos Aires would have approximately 320,000 children aged 0–14, 108,000 women of reproductive age (15–49), and 170,000 women aged 50 and older. Figure 3 shows a significant increase in births (0-14, from 120k to 320k) reflects the strong fecundity of the initial 15-49 group. As survivors age, the reproductive cohort decreases to 108k, while the 50+ bar increases to 170k.

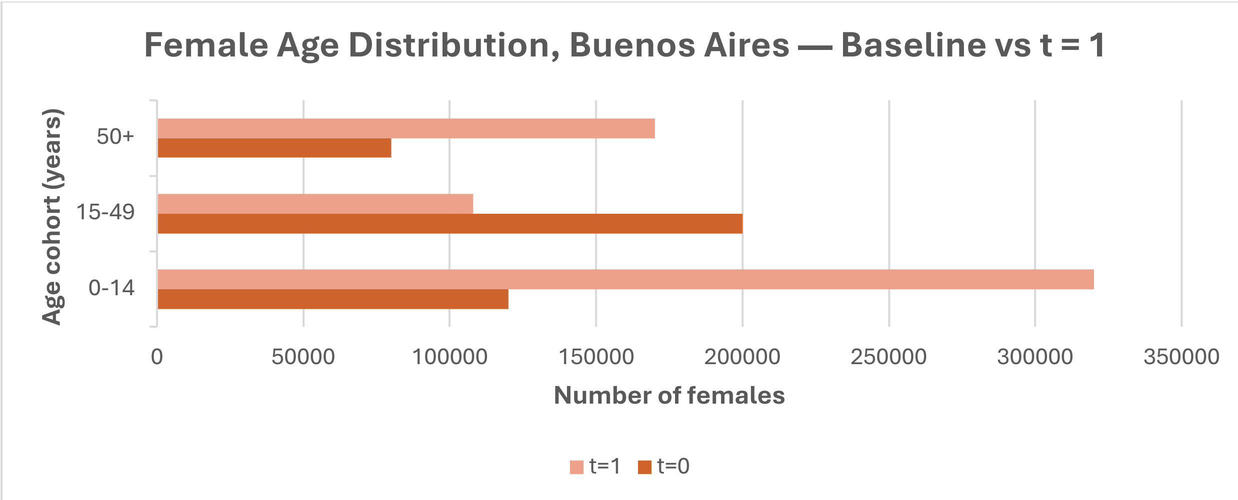


Figure 4

Projecting Future Population Changes

Building on the previous example, the Leslie matrix enables us to predict population fluctuations over multiple time periods. The technique is iteratively multiplying the current population vector with the Leslie matrix, which updates the number of people in each age group based on fertility and survival rates. Let's run the identical scenario for two more time periods, $N(2)$ and $N(3)$, to see how the population structure changes with time.

Second Time Period

In the second time step, the population vector is calculated as $N\ (2) = A * N(1)$, where $N(1)$ is the result from the previous time step.

$$N\ (2) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix} . \begin{bmatrix} 320{,}000 \\ 108{,}000 \\ 170{,}000 \end{bmatrix}$$

The computations for each age class are as follows.

Newborns (0–14 years):

$$N_0 = (0) . (320{,}000) + (1.4) . (108{,}000) + (0.5) . (170{,}000)$$

$$N_0 = 0 + 151{,}200 + 85{,}000$$

$$N_0 = 236{,}200$$

Survivors from age group 0–14 to 15–49:

$$N_1 = (0.9) . (320{,}000) + (0) . (108{,}000) + (0) . (170{,}000)$$

$$N_1 = 288{,}000 + 0 + 0$$

$$N_1 = 288{,}000$$

Survivors from age group 15–49 to 50+:

$$N_2 = (0) . (320{,}000) + (0.85) . (108{,}000) + (0) . (170{,}000)$$

$$N_2 = 0 + 91{,}800 + 0$$

$$N_2 = 91{,}800$$

The updated population vector becomes:

$$N\ (2) = \begin{bmatrix} 236{,}200 \\ 288{,}000 \\ 91{,}800 \end{bmatrix}$$

Interpretation After Two Time Steps

At this point, Buenos Aires would have 236,200 children aged 0 to 14, 288,000 women of reproductive age, and 91,800 women aged 50 and up. Figure 4 reflect how initial baby boom feeds into the reproductive bar (now 288 k), resulting in births falling to 236 k even as the mid-age cohort rebounds. The 50+ bar lowers to around 92k after its first spike, indicating the system's early oscillation.

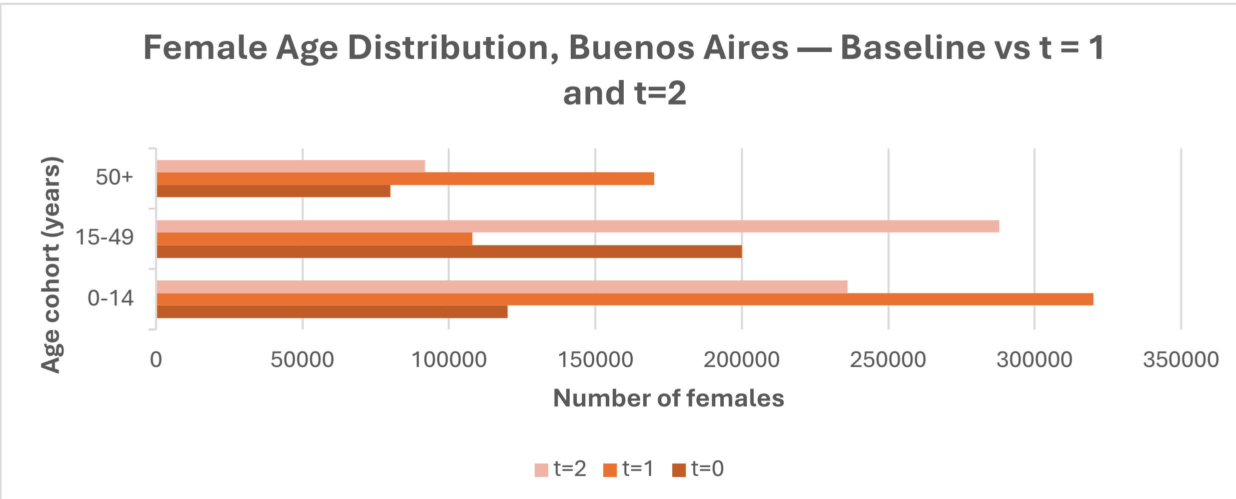


Figure 5

Third Time Period

In the third time step, the population vector is calculated as $N\ (3) = A * N(2)$, where $N(2)$ is the result from the previous time step.

$$N\ (3) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix} . \begin{bmatrix} 236{,}200 \\ 288{,}000 \\ 91{,}800 \end{bmatrix}$$

The computations for each age class are as follows.

Newborns (0–14 years):

$$N_0 = \ (0)\,.(236{,}200) + (1.4)\,.(288{,}000) + (0.5)\,.(91{,}800)$$

$$N_0 = 0 + 403{,}200 + 45900$$

$$N_0 = 449{,}100$$

Survivors from age group 0–14 to 15–49:

$$N_1 = (0.9) \,.\, (236{,}200) + (0) \,.\, (288{,}000) + (0) \,.\, (91{,}800)$$

$$N_1 = 212{,}580 + 0 + 0$$

$$N_1 = 212{,}580$$

Survivors from age group 15–49 to 50+:

$$N_2 = (0) \,.\, (236{,}2000) + (0.85) \,.\, (288{,}000) + (0) \,.\, (91{,}800)$$

$$N_2 = 0 + 244{,}800 + 0$$

$$N_2 = 244{,}800$$

The updated population vector becomes:

$$N\ (3) = \begin{bmatrix} 449{,}100 \\ 212{,}580 \\ 244{,}800 \end{bmatrix}$$

Interpretation After Three Time Steps

After three time periods, the population consists of 449,100 children aged 0-14, 212,580 women of reproductive age, and 244,800 women aged 50 and up. At t=2, the increased reproductive pool leads to a second, larger baby wave (0-14 rises to 449 k). Figure 5 shows how the elderly population grows to around 245k, indicating a long-term trend toward an older age structure, regardless of periodic fluctuations.

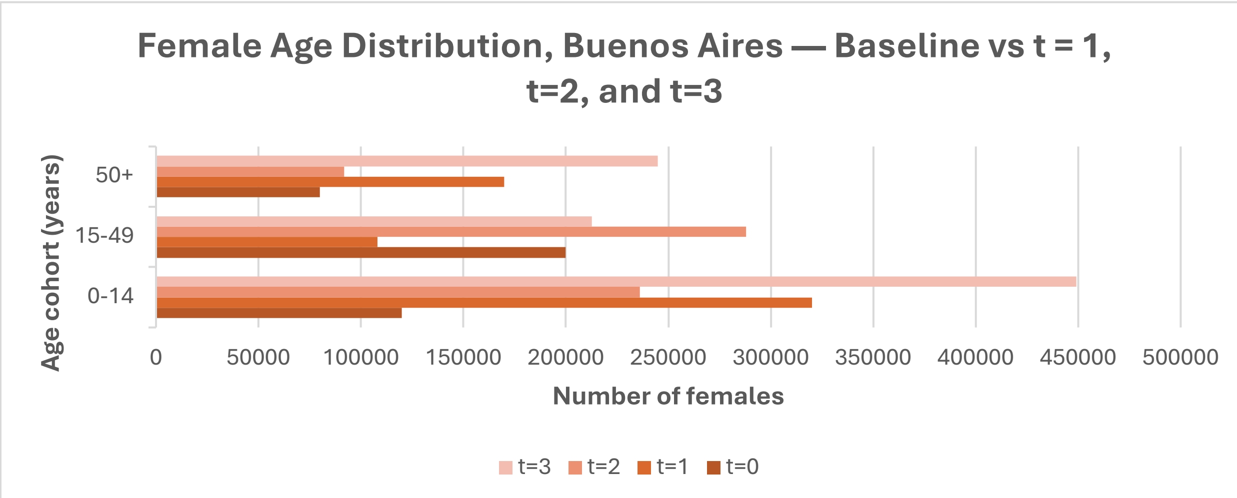


Figure 6

Dominant Eigenvalue of This Example and Its Interpretation

The long-term behavior of the population is linked to the eigenvalues and eigenvectors of the Leslie matrix A (Caswell, 1989). An eigenvalue λ and its corresponding eigenvector v satisfy the equation:

$$Av = \lambda v$$

This equation can be rearranged as:

$$Av - \lambda v = 0$$

$$(A - \lambda I)v = 0$$

where I is the identity matrix. For this equation to have a nonzero solution (i.e., for $v \neq 0$), the matrix $A - \lambda I$ must be singular. A matrix is singular if and only if its determinant is zero (Lay, 2012). Therefore, we set up the characteristic equation:

$$det(A - \lambda I) = 0$$

For our Leslie matrix, we have:

$$A - \lambda I = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix} + \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1.4 & 0.5 \\ 0.9 & -\lambda & 0 \\ 0 & 0.85 & -\lambda \end{bmatrix}$$

Let's compute the determinant of: $\begin{bmatrix} -\lambda & 1.4 & 0.5 \\ 0.9 & -\lambda & 0 \\ 0 & 0.85 & -\lambda \end{bmatrix}$

Using the rule for 3×3 determinants:

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 \\ 0.85 & -\lambda \end{vmatrix} - 1.4 \begin{vmatrix} 0.9 & 0 \\ 0 & -\lambda \end{vmatrix} + 0.5 \begin{vmatrix} 0.9 & -\lambda \\ 0 & 0.85 \end{vmatrix}$$

Now compute each 2×2 determinant:

$$\begin{vmatrix} -\lambda & 0 \\ 0.85 & -\lambda \end{vmatrix} = [(-\lambda)x(-\lambda)] - [(0)x(0.85)] = (-\lambda)^2$$

$$\begin{vmatrix} 0.9 & 0 \\ 0 & -\lambda \end{vmatrix} = [(0.9)x(-\lambda)] - [(0)x(0)] = -0.9\lambda$$

$$\begin{vmatrix} 0.9 & -\lambda \\ 0 & 0.85 \end{vmatrix} = [(0.9)x(0.85)] - [(-\lambda)x(0)] = 0.765$$

Plug these back into the determinant calculation:

$$\det(A - \lambda I) = -\lambda[(-\lambda)^2] - 1.4[(-0.9\lambda)] + 0.5(0.765) = 0$$

$$\det(A - \lambda I) = -\lambda^3 + 1.26\lambda + 0.3825 = 0$$

Multiplying both sides by -1 (to make the cubic term positive) gives:

$$\lambda^3 - 1.26\lambda - 0.3825 = 0$$

By solving[2] this cubic equation, we find that the largest real solution, called the dominant eigenvalue, is approximately:

$$\lambda \approx 1.2512747$$

This value means that under the given fertility and survival rates, the total population increases by about 25.1% per time step.

Total Population at Each Time Step

We have the population vectors for $N(0), N(1), (2), and\ N(3)$. We add up the entries in the corresponding vector to determine the total population at each stage.

$$N(0) = \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix}$$

[2] To obtain this result, I used Wolfram Alpha to solve the equation, as the algebraic process is lengthy.

$$T(0) = 120{,}000 + 200{,}000 + 80{,}000 = 400{,}000$$

$$N\ (1) = \begin{bmatrix} 320{,}000 \\ 108{,}000 \\ 170{,}000 \end{bmatrix}$$

$$T(1) = 320{,}000 + 108{,}000 + 170{,}000 = 598{,}000$$

$$N\ (2) = \begin{bmatrix} 236{,}200 \\ 288{,}000 \\ 91{,}800 \end{bmatrix}$$

$$T(2) = 236{,}200 + 288{,}000 + 91{,}800 = 616{,}000$$

$$N\ (3) = \begin{bmatrix} 449{,}100 \\ 212{,}580 \\ 244{,}800 \end{bmatrix}$$

$$T(3) = 449{,}100 + 212{,}580 + 244{,}800 = 906{,}480$$

Ratios of Successive Total Populations

$$R(1) = \frac{T(1)}{T(0)} = \frac{598{,}000}{400{,}000} = 1.495$$

$$R(2) = \frac{T(2)}{T(1)} = \frac{616{,}000}{598{,}000} = 1.0301$$

$$R(3) = \frac{T(3)}{T(2)} = \frac{906{,}480}{616{,}000} = 1.4715$$

Observing the Ratios vs. the Dominant Eigenvalue

Based on the Leslie matrix's characteristic equation, we estimated the dominant eigenvalue to be around 1.2512747. The growth rate, or the ratio of successive total populations, tends to cluster around this dominant eigenvalue over time as the population vector iterates through numerous time steps (Caswell, 1989). Only three iterations have been calculated. The population structure is still far from reaching its "stable" proportions at this early stage of the projection. The growth ratios may therefore fluctuate above or below the ultimate long-term rate.

Long-Term Behavior of the Leslie Matrix Model

To evaluate the long-term forecasting power of the Leslie matrix model for the Buenos Aires population, we compute how the total population evolves over various time steps. We generate new population vectors by repeatedly multiplying the Leslie matrix by $N(0) = \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix}$. For each pair of consecutive time steps, we compute the ratio of total populations, denoted as: $R(T) = \frac{T(t)}{T(t-1)}$.

The purpose is to see if these ratios stabilize around the dominant eigenvalue, which was previously estimated as λ=1.2512747.

Let's walk through the calculations:

Time Steps 7 and 8

$$(A^7) * N(0) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}^7 x \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix} = \begin{bmatrix} 987482.3400000001 \\ 630896.2830000002 \\ 502002.5625 \end{bmatrix}$$

$$(A^{8}) * N(0) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}^{8} x \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix} = \begin{bmatrix} 1134256.0774500002 \\ 888734.106 \\ 536261.84055 \end{bmatrix}$$

Ratio of Successive Totals:

$$R(8) = \frac{T(8)}{T(7)} = \frac{1134256.0774500002 + 888734.106 + 536261.84055}{987482.3400000001 + 630896.2830000002 + 502002.5625} \approx 1.207472$$

The ratio is improving toward λ, but it’s still somewhat below the expected long-term rate. This indicates the population hasn’t fully stabilized yet.

Time Steps 14 and 15

$$(A^{14}) * N(0) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}^{14} x \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix} = \begin{bmatrix} 4492305.472493753 \\ 3274924.404194199 \\ 2184181.528614468 \end{bmatrix}$$

$$(A^{15}) * N(0) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}^{15} x \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix} = \begin{bmatrix} 5676984.930179112 \\ 4043074.9252443784 \\ 2783685.743565069 \end{bmatrix}$$

Ratio of Successive Totals:

$$R(15) = \frac{T(15)}{T(14)}$$

$$= \frac{4492305.472493753 + 3274924.404194199 + 2184181.528614468}{5676984.930179112 + 4043074.9252443784 + 2783685.743565069} \approx 1.2565$$

The ratio is now extremely close to the dominant eigenvalue λ. We are beginning to see the effects of stabilization and long-term growth.

Time Steps 24 and 25

$$(A^{24}) * N(0) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}^{24} x \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix} = \begin{bmatrix} 42497753.99828328 \\ 30585901.415088896 \\ 20759929.6303493 \end{bmatrix}$$

$$(A^{25}) * N(0) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}^{25} x \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix} = \begin{bmatrix} 53200226.79629911 \\ 38247978.59845496 \\ 25998016.20282556 \end{bmatrix}$$

Ratio of Successive Totals:

$$R(25) = \frac{T(25)}{T(24)}$$

$$= \frac{53200226.79629911 + 38247978.59845496 + 25998016.20282556}{42497753.99828328 + 30585901.415088896 + 20759929.6303493} \approx 1.2512747$$

This result is almost identical to the dominant eigenvalue. It confirms that the model's predictions have fully stabilized and long-term population growth is now governed by $\lambda \approx 1.2512747$

Graphical Illustration of Convergence

To visualize how the growth ratio $R(t) = \frac{T(t)}{T(t-1)}$ behaves over successive iterations, Figure 7 below displays population ratios across the first 30 time steps. The black points represent the ratios calculated above using matrix multiplication, while the orange points were generated in Excel to fill in the remaining values and visualize the complete trend.

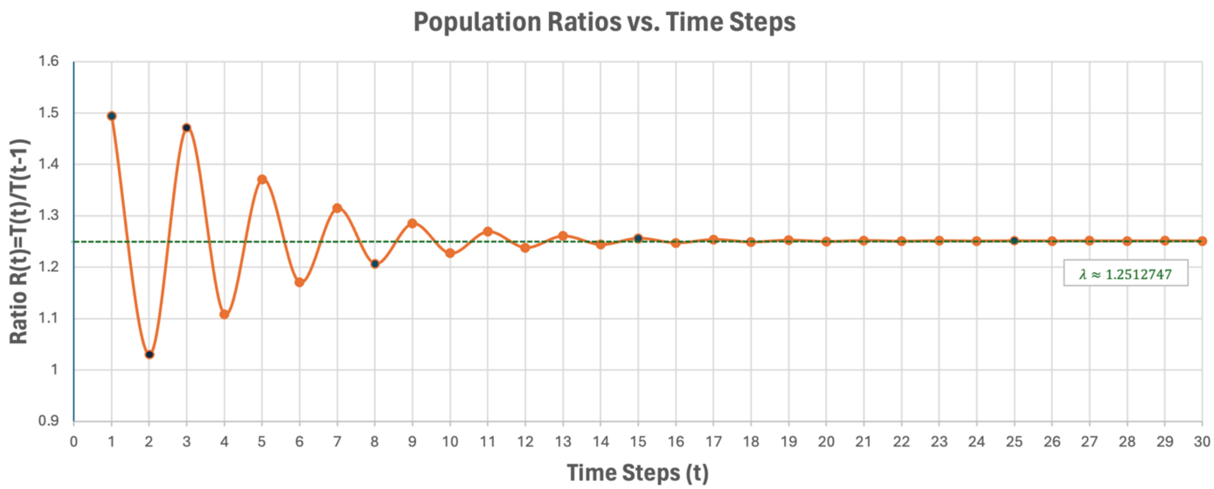


Figure 7

As illustrated in the graph, the early ratios (R(1), R(2), R(3)) vary greatly. Some values are far higher than 1.4, while others fall closer to 1.0. This fluctuation represents the impact of the initial population distribution, which has yet to settle into the steady-state structure indicated by the Leslie matrix.

As more iterations are performed, the ratios start to stabilize. For example, R(25) indicate that the system is increasingly reaching a regular growth pattern. This pattern roughly matches with the theoretical dominant eigenvalue (λ). The values begin to cluster tightly around 1.25 after around 20 time steps. This trend indicates that, despite initial fluctuation, the Leslie matrix model eventually converges to a predictable, consistent growth rate.

Finding the Corresponding Eigenvector

Next, we find the eigenvector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ associated with $\lambda \approx 1.251274$. We solve:

$$(A - \lambda I) = 0$$

Substitute $\lambda \approx 1.255$ into the matrix:

$$A - 1.251274I = 0$$

$$\begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix} - 1.251274 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1.251274 & 1.4 & 0.5 \\ 0.9 & -1.251274 & 0 \\ 0 & 0.85 & -1.251274 \end{bmatrix} = 0$$

This yields the system of linear equations:

$$\begin{cases} -1.251274\text{x} + 1.4\text{y} + 0.5\text{z} = 0 & (1) \\ -0.9\text{x} - 1.251274\text{y} + 0\text{z} = 0 & (2) \\ 0x \ + \ 0.85y \ - 1.251274\text{z} = 0 & (3) \end{cases}$$

Step 1: Solve Equation (2) for x

$$-0.9\text{x} - 1.251274\text{y} + 0\text{z} = 0 \Rightarrow 0.9x = 1.251274\, y \Rightarrow x = \frac{1.251274}{0.9}\, y \Rightarrow x \approx 1.3903\, y$$

Step 2: Solve Equation (3) for z

$$0x \ + \ 0.85y \ - 1.251274\text{z} = 0 \ \Rightarrow \ -1.251274\text{z} = -0.85y \ \Rightarrow z = \frac{-0.85}{-1.251274} y$$

$$\Rightarrow z \approx 0.679308\, y$$

Thus, an eigenvector is proportional to:

$$v = \begin{bmatrix} 1.3903\, y \\ \text{y} \\ 0.679308\, y \end{bmatrix}$$

For simplicity, set $y = 1$. Then:

$$v = \begin{bmatrix} 1.3903 \\ 1 \\ 0.679308 \end{bmatrix}$$

Step 3: Normalize the Eigenvector

To interpret v as a stable age distribution, we normalize it so that the sum of its components equals 1:

$$1.39034 + 1 + 0.67930 = 3.06964$$

The normalized eigenvector is:

$$v = \begin{bmatrix} \frac{1.3903}{3.06964} \\ \frac{1}{3.06964} \\ \frac{0.679308}{3.06964} \end{bmatrix} \approx \begin{bmatrix} 0.45292 \\ 0.325771 \\ 0.221299 \end{bmatrix}$$

This means that in the long run:

- Age Group 1 (Children 0–14 years): Approximately 45.2% of the population
- Age Group 2 (Reproductive Age 15–49 years): Approximately 32.5% of the population
- Age Group 3 (Post-Reproductive 50+ years): Approximately 22.1% of the population

These percentages represent the stable age distribution for the Buenos Aires scenario.

Confirming via $\boldsymbol{A^{25}.N(0)}$

Even by multiplying $\boldsymbol{A^{25}}$ by the initial population vector $\boldsymbol{N(0)}$, we see similar proportions emerge.

$$(A^{25}) * N(0) = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}^{25} x \begin{bmatrix} 120{,}000 \\ 200{,}000 \\ 80{,}000 \end{bmatrix} = \begin{bmatrix} 53200226.79629911 \\ 38247978.59845496 \\ 25998016.20282556 \end{bmatrix}$$

Summing the entries gives a total population:

$$T(25) = 53200226.79629911 + 38247978.59845496 + 25998016.20282556$$

$$T(25) = 117{,}446{,}221.59757963$$

Dividing each component by that total:

$$\frac{53200226.79629911}{117{,}446{,}221.59757963} = 0.452878$$

$$\frac{38247978.59845496}{117{,}446{,}221.59757963} = 0.325777$$

$$\frac{25998016.20282556}{117{,}446{,}221.59757963} = 0.221345$$

These ratios are nearly identical to the normalized eigenvector $v \approx \begin{bmatrix} 0.45292 \\ 0.325771 \\ 0.221299 \end{bmatrix}$, showing that by the 25th iteration, the population vector already reflects the stable age distribution quite closely.

Relating the Leslie Matrix to Actuarial Science

Demographic projections are predictions about how a population will evolve over time. The Leslie matrix model, which employs age-specific fertility and survival rates, contributes to the creation of these estimates (Keyfitz & Caswell, 2005). Understanding these estimates is critical for actuaries because many insurance and pension risks are based on future demographic shifts.

Actuaries utilize demographic data to anticipate how many people will be alive in the future. For example, in life insurance, forecasting how many policyholders will live to a specific age helps determine premiums and reserves (Lee & Carter, 1992; Bowers et al., 1997). The Leslie matrix model can provide a clear picture of population structure over time by demonstrating how groups of people transition from one age category to the next. This information is useful in estimating the likelihood of claims and the amount of money that should be set aside to pay for future payouts.

Using the Leslie matrix model, actuaries can improve risk assessment. They can model numerous scenarios by modifying fertility and survival rates, which are influenced by public health improvements, lifestyle changes, and unexpected events such as pandemics (Koslucher, 2016; Klugman, Panjer, & Willmot, 2012). This method enables actuaries to estimate the impact of these changes on the future population. As a result, actuaries can better anticipate for variations in mortality and morbidity rates, lowering uncertainty in their estimates (Billig & Gallop, 2014).

Implications for Pension and Life Insurance Models

The Leslie matrix model's results have direct implications for pension systems and life insurance plans. Pensions and life insurance rely substantially on accurate mortality and survival rates (Pollard, 1973).

One of the most difficult aspects of pension models is predicting how long retirees will live. The Leslie matrix allows you to forecast the future age distribution of a population (Leslie, 1948). Actuaries can anticipate how many individuals will retire and how long they will live based on the steady age distribution. This information is critical for determining contribution rates and ensuring that pension plans are solvent over time. If the population lives longer than predicted, pension plans may need to change payouts or raise contributions to meet long-term expenses (Bowers et al., 1997).

In the case of life insurance, the model can help anticipate how many claims will be submitted in the future. Life insurance firms utilize mortality tables to calculate the chance of a policyholder's death at each age. The Leslie matrix model refines these estimations by accounting for the population's whole structure (Keyfitz & Caswell, 2005). This makes pricing policies more precise. If the model predicts that a greater proportion of the population will live to an extended age, insurers may need to raise premiums or change benefits to account for the higher risk of longer lifespans (Klugman, Panjer, & Willmot, 2012).

Furthermore, the Leslie matrix model may help stress test insurance portfolios. Actuaries can construct multiple situations by adjusting the matrix's input parameters, such as fertility and survival rates (Billig & Gallop, 2014; Xie et al., 2020). For example, if an epidemic or a significant decline in survival rates happens, the model will demonstrate how the age distribution changes. Insurers can then assess how these changes will affect their liabilities and adapt their strategy accordingly (Bowers et al., 1997).

Advantages of Using Matrix Models in Actuarial Forecasting

Matrix models, such as the Leslie matrix, have significant advantages over classic actuarial forecasting approaches. One of the key advantages is their ability to simplify complicated demographic information. Instead of evaluating many individual data points, actuaries may construct a matrix to describe the population's general behavior using only a few important numbers (fertility and survival rates). This simplification makes it simpler to see trends and analyze the population's long-term behavior.

Another key benefit is the repetitive nature of matrix multiplication. With each multiplication, the model predicts the population one time period ahead (Leslie, 1945). This repeating approach enables actuaries to anticipate several periods in the future using a single, consistent method. As the predictions go, the age distribution frequently stabilizes, which aids in analyzing long-term patterns. Understanding the stable age distribution is very beneficial for developing long-term financial solutions since it gives a clear aim for future population structure.

Matrix models also provide flexibility. Actuaries may easily alter the input variables to simulate various scenarios and determine how changes in survival or fertility rates effect the entire population. This adaptability is crucial in actuarial science, where changes in public policy, economic situations, or unanticipated occurrences can have a large influence on population patterns.

Furthermore, a greater comprehension of the dynamics of population changes is made possible by the mathematical foundations of matrix models. Actuaries can evaluate the population growth rate by using ideas such as eigenvalues and eigenvectors, which are directly derived from the Leslie matrix. For instance, the total rate of population growth or decline is represented by the dominant eigenvalue (Caswell, 1989). Actuaries can improve the accuracy of their forecasts and make better decisions by contrasting this rate with the assumptions made in insurance and pension models.

Implications for Pension and Life Insurance Models: Buenos Aires Example

Recall the scenario in Buenos Aires, where the population is split into three age groups:

- Age Group 1: Children aged 0 to 14 years = 120,000
- Age Group 2: 15 to 49 years (reproductive age) = 200,000
- Age Group 3: 50+ years old (post-reproductive age) = 80,000

The Leslie matrix used in this model is defined as:

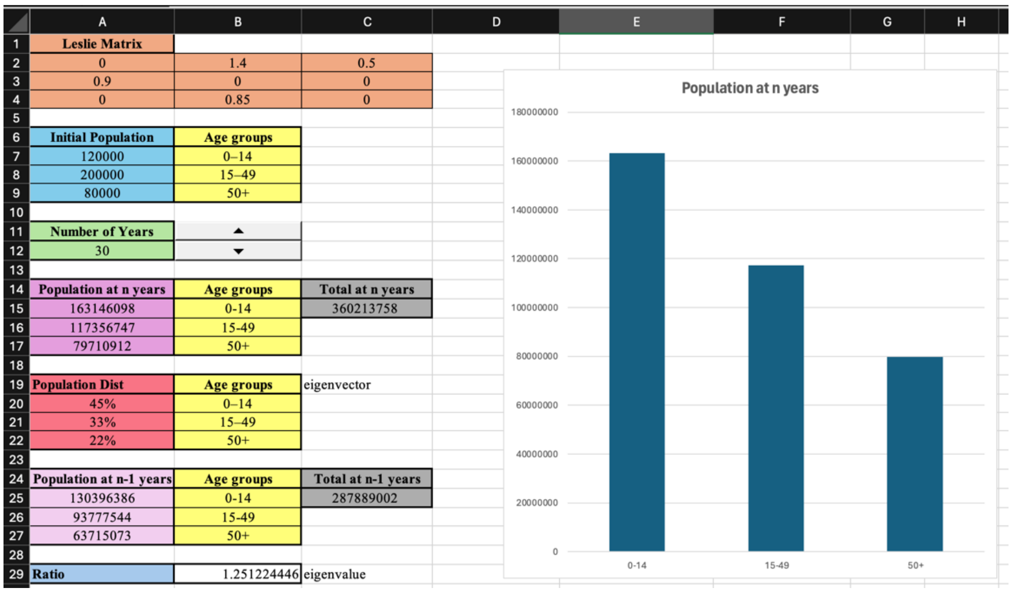
$$A = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.85 & 0 \end{bmatrix}$$

Projection after 30 Time Steps:

The projection is extended to 30 time steps, rather than examining short-term shifts across one, two, and three time steps. The population vector approaches a very stable age distribution during 30 iterations of the Leslie matrix multiplication. The approximate projections derived from an Excel-based computations are as follows:

- Age Group 1 (0–14 years): ≈ 163,146,098 individuals (~45% of total)

- Age Group 2 (15–49 years): ≈ 117,356,747 individuals (~33% of total)
- Age Group 3 (50+ years): ≈ 79,710,912 individuals (~22% of total)



The estimated total population is 360,213,758 people. Furthermore, the dominant eigenvalue of the Leslie matrix is extremely near to the ratio of successive total populations (i.e., T(30)/T(29)), which has converged to about 1.2513. At each time step, this dominant eigenvalue suggests a growth rate of approximately 25.13%.

From an actuarial perspective:

What the 30-year projection looks like:

	t = 0	t = 30	Growth-factor
Children (0–14)	120,000	163,146,098	$\frac{163{,}146{,}098}{120{,}000}(100) = 135955\%$
Working age (15–49)	200,000	117,356,747	$\frac{117{,}356{,}747}{200{,}000}(100) = 58678\%$

Older adults (50 +)	80,000	79,710,912	$\frac{79,710,912}{80,000}(100) = 99639\%$
Total population	400,000	360,213,757	$\frac{360,213,757}{400,000}(100) = 90053\%$

Pensions

Indicator	t = 0	t = 30
Retiree-to-worker ratio	$\frac{80,000}{200,000} = 0.4$	$\frac{79,710,912}{117,356,747} = 0.68$
Overall dependency ratio	$\frac{120,000 + 80,000}{200,000} = 1$	$\frac{163,146,098 + 79,710,912}{117,356,747} = 2.07$

As the population changes, the pension system will become increasingly top-heavy. Thirty years ago, every 100 workers supported approximately 40 retirees; now, the same 100 workers will support 68. To maintain benefits, a pay-as-you-go plan would need to collect almost 70% more from each employee, as there are nearly one retiree for every 1.5 workers. When retirees and children are combined, the number of dependents exceeds the working-age population by around 2 to 1. In other words, each worker now has twice as many people on their shoulders as they had at the beginning of the projection.

Actuaries have three simple techniques to help ease this burden. Raising contribution rates, such as increasing payroll deductions from 10% to 17%, can help close the difference. Raising retirement age from 62 to 67 for longer contributions and shorter benefit periods. Finally, they can create or increase funded tiers, allowing investment returns (rather than current employee pay) to cover a portion of future pension expenditures.

Life Insurance

Regarding life insurance, the situation is dramatic but no longer mysterious because of the model. Thirty years later, the number of older citizens who are most likely to file death claims has increased from about 80,000 at the beginning of the projection to over 79.7 million. Even if that is a significant increase, their population proportion remains constant at 22%.

Actuaries can now anticipate the number of deaths annually with much higher confidence because this fraction no longer fluctuates. This means that the risk of people living surprisingly long lives, often known as longevity risk, can now be measured rather than estimated.

How should this information be handled by insurers?

First, they can represent the progressive growth in risk by evenly spreading price increases across the age range rather than slamming customers with a huge premium surge on their fiftieth birthday. Second, they need to increase their reserves because, at a normal payout of $100,000 per policy, the cost of claims each year rises from a few million dollars now to several billion dollars when the older group ages. Last but not least, a steady 22% senior share facilitates the creation of annuity-style products whose long-term cash flows align with those predicted future claims, allowing investment returns to lessen the load rather than shifting the entire cost to policyholders.

Pandemic scenario — reduced survival rates

Recall the scenario in Buenos Aires, where the population is split into three age groups:

- Age Group 1: Children aged 0 to 14 years = 120,000
- Age Group 2: 15 to 49 years (reproductive age) = 200,000

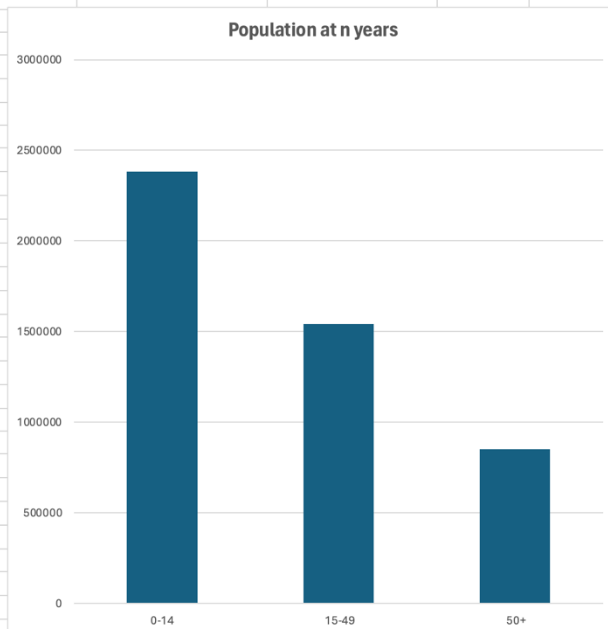
- Age Group 3: 50+ years old (post-reproductive age) = 80,000

To see how the same population in Buenos Aires would alter if a severe, prolonged pandemic hit, we maintain fertility at the same level but reduce survival chances by about 25%:

$$A_{pandemic} = \begin{bmatrix} 0 & 1.4 & 0.5 \\ 0.7 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$$

Projection after 30 Time Steps:

	A	B	C
1	**Leslie Matrix**		
2	0	1.4	0.5
3	0.7	0	0
4	0	0.6	0
5			
6	**Initial Population**	**Age groups**	
7	120000	0–14	
8	200000	15–49	
9	80000	50+	
10			
11	**Number of Years**	▲	
12	30	▼	
13			
14	**Population at n years**	**Age groups**	**Total at n years**
15	2383557	0-14	4777676
16	1541490	15-49	
17	852628	50+	
18			
19	**Population Dist**	**Age groups**	eigenvector
20	50%	0–14	
21	32%	15–49	
22	18%	50+	
23			
24	**Population at n-1 years**	**Age groups**	**Total at n-1 years**
25	2202129	0-14	4411359
26	1421047	15-49	
27	788183	50+	
28			
29	**Ratio**	1.083039375	eigenvalue



Pandemic Scenario with Reduced Survival Rates

The baseline population in this other scenario still includes 200,000 people of working age (15–49), 80,000 senior adults (50+), and 120,000 children (0–14). The transition probability from one age group to the next have been decreased, though, as the survival rates have been reduced to represent pandemic conditions. The population increases from 400,000 to 4,777,676 after 30 time steps. Compared to the non-pandemic baseline, when the population increased by around 900 times, this growth is much slower, although still being significant.

The change is particularly noticeable when the major age groups are broken down. From 120,000 to 2,383,557 children (0–14). The working-age population (15–49) has increased from 200,000 to 1,541,490. In comparison, the number of elderly individuals (those over 50) increases from 80,000 to 852,628. Older adults do rise considerably in the pandemic scenario, but not nearly as dramatically as they did in the baseline, when survival rates were greater.

	t = 0	t = 30	Growth-factor
Children (0–14)	120,000	2,383,557	$\frac{2{,}383{,}557}{120{,}000}(100) = 1986\%$
Working age (15–49)	200,000	1,541,490	$\frac{1{,}541{,}490}{200{,}000}(100) = 771\%$
Older adults (50 +)	80,000	852,628	$\frac{852{,}628}{80{,}000}(100) = 1066\%$
Total population	400,000	4,777,676	$\frac{4{,}777{,}676}{400{,}000}(100) = 1194\%$

Pensions

Indicator	t = 0	t = 30 (Pandemic Scenario)	t = 30 (Normal Baseline)
Retiree-to-worker ratio	$\frac{80{,}000}{200{,}000} = 0.4$	$\frac{852{,}628}{1{,}541{,}490} = 0.55$	$\frac{79{,}710{,}912}{117{,}356{,}747} = 0.68$
Overall dependency ratio	$\frac{120{,}000 + 80{,}000}{200{,}000} = 1$	$\frac{2{,}383{,}557 + 852{,}628}{1{,}541{,}490} = 2.1$	$\frac{163{,}146{,}098 + 79{,}710{,}912}{117{,}356{,}747} = 2.07$

There were 40 retirees for every 100 employees at the beginning. After thirty time steps, the ratio is roughly 0.55, or 55 retirees for every 100 employees. Even if that is less than the baseline pandemic-free ratio of 0.68, it still means that in order to maintain benefits at the same level, today's contribution rate must increase, or retirement ages must be raised. The overall dependency ratio increases from 1.0 to about 2.10 when children are included. To put it simply, each employee now provides for somewhat more than two dependents rather than just one. The plan must fund both education-style expenses (public services for the young) and pension payments for the elderly because the burden is distributed about equally between retirees and children.

The toolkit for stabilizing the pension plan remains the same as in the baseline situation, but the dial can be turned down. Minor payroll deductions decrease compared to the normal baseline (from 17% to 12%). Subtracting 4 years to the retirement age (compared to the baseline scenario) from 67 to 63 since people may not live as long, this would still reduce the retiree-to-worker ratio and increases the payout horizon for contributors. Finally, building emergency savings earlier to prepare for sudden changes in survival rates or future pandemics, lowering the need for significant tax increases down the road.

Life Insurance

Lower survival affects both the timing and the size of claims. The elder cohort is "only" 10 times larger (about 0.85 million), hence the end claim pool is significantly lower than under the baseline. However, mortalities occur earlier: reserves must cover a steeper payout curve in the first few decades, after which longevity risk is lessened as fewer people reach extreme ages.

In order to collect enough to pay the future rise in claims without shocking customers, insurers can adjust to the pandemic profile by smoothing premiums, raising rates gradually between ages 40 and 65 instead of imposing a dramatic sudden increase at 50. At the same time, companies can set aside a specific "pandemic reserve," about 10 to 15 percent of premium income in the first ten years, to protect against early cash outflows during the peak of excess mortality. Last but not least, changing the product mix to include conservatively priced annuities and five- to ten-year term-life insurance plans matches contract lengths with front-loaded risk and acknowledges that a far smaller percentage of policyholders would live to extreme old age.

When compared to the baseline, the pandemic scenario generally lessens the long-term strain, although it causes financial pressure to increase over time. Therefore, actuaries need to rebalance cash flows: life insurers need to be prepared for an early, concentrated surge of claims, whereas pensions need a moderate, continuous adjustment.

Conclusion

This thesis demonstrated the effectiveness of the Leslie matrix model as a tool for analyzing population dynamics over time, particularly in actuarial applications. We were able to investigate how fertility and survival rates influence both short-term fluctuations and long-term population structure convergence by modeling the hypothetical population of Buenos Aires.

The total population grows at a pace that converges to the dominant eigenvalue, approximately $\lambda = 1.2512747$, according to our analysis of eigenvalues and matrix multiplication. Growth ratios in the early iterations demonstrated significant oscillation, which was reflective of the original population distribution's influence. However, their oscillations decreased as the time steps increased. By the 25th time step, the relative age distribution and the ratio of succeeding total populations nearly matched the dominant eigenvalue and corresponding eigenvector predictions.

This convergence was further supported by a graphic analysis of the growth ratios, which showed that the long-term growth rate was clustered around calculated points (black) and Excel-generated estimates (orange). Despite initial fluctuation, the system consistently settles into a steady growth pattern and stable age structure over time, as demonstrated by the visualizations, demonstrating the Leslie matrix model's predictive capacity.

Lastly, we demonstrated how models such as the Leslie matrix could guide policy decisions about insurance, pensions, and demographic planning by relating these mathematical insights to actuarial science. These techniques provide actuaries with a structured, quantitative framework to predict future trends and create sustainable financial systems in an era characterized by population aging, longevity risk, and economic unpredictability.

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